SUMMARY NOTES ON A RASCH MODEL FOR LIKERT SCALES

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1. Motivation from the dichotomous model.

(a) Consider one person v responding to items $i = 1, K$.
In the dichotomous response situation

$$p_{vi}^{(0)} = p\{x_{vi} = 0\} = 1/(1+e^{\eta_v + \sigma_i})$$
$$p_{vi}^{(1)} = p\{x_{vi} = 1\} = e^{\eta_v+\sigma_i}/(1+e^{\eta_v + \sigma_i})$$

$$\therefore \quad p_{vi}^{(1)}/p_{vi}^{(0)} = e^{\eta_v+\sigma_i}$$
$$\log(p_{vi}^{(1)}/p_{vi}^{(0)}) = l_i = \eta_v + \sigma_i$$

(b) To simplify our motivation for the polychotomous model, suppose in the first instance that for $v = 1, N$, all η_v are the same - say $\eta_v = 0$. Later we reintroduce the subject parameter.
In that case we have

$$\boxed{\begin{aligned} p_i^{(0)} &= 1/(1+e^{\sigma_i}) \\ p_i^{(1)} &= e^{\sigma_i}/(1+e^{\sigma_i}) \end{aligned}}$$

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$$p_i^{(1)}/p_i^{(0)} = e^{\sigma_i}$$
and $$l_i = \sigma_i$$

(c) For a given set of responses to the items, we could estimate each σ_i by taking the proportion of subjects who get each item incorrect and each item correct giving $\hat{p}_i^{(0)}, \hat{p}_i^{(1)}$. Then

$$\log(\hat{p}_i^{(1)}/\hat{p}_i^{(0)}) = \hat{l}_i = \hat{\sigma}_i .$$

However, we usually require some relative relation among the item parameters. So we may set

$$\hat{\sigma}_i = \hat{\mu} + \hat{\theta}_i$$

where now $\sum_{i=1}^{k} \hat{\theta}_i = 0$.

(a) A diagramatic representation of these ideas is shown below

	0	1	→	$p_i^{(1)}/p_i^{(0)}$	σ_i	μ
1	$p_1^{(0)}$	$p_1^{(1)}$		$p_1^{(1)}/p_1^{(0)}$	σ_1	θ_1
2	$p_2^{(0)}$	$p_2^{(1)}$		$p_2^{(1)}/p_2^{(0)}$	σ_2	θ_2
⋮	⋮	⋮		⋮	⋮	⋮
i	$p_i^{(0)}$	$p_i^{(1)}$		$p_i^{(1)}/p_i^{(0)}$	σ_i	θ_i
⋮	⋮	⋮		⋮	⋮	⋮
K	$p_K^{(0)}$	$p_K^{(1)}$		$p_K^{(1)}/p_K^{(0)}$	σ_K	θ_K

For example:

					0.14
1	·20	·80	4.00	1.39	1.25
2	·25	·75	3.00	1.10	0.96
3	·50	·50	1.00	0.00	-0.14
4	·6̇6̇	·3̇3̇	·50	-0.70	-0.84
5	·80	·20	·25	-1.10	-1.24
				$\sum_i \sigma_i = 0.90$	$\sum_i \theta_i = 0.0$

Notice that the set up is much like that of the one way ANOVA with μ corresponding to the grand mean and θ_i corresponding to the deviation about the mean. i.e. the group effect. In the above case, other techniques are used to estimate the item parameters – we are doing it this way here in order to get a feel for what is happening.

Notice also that although there are two response categories, we have and need only one item parameter. This is because we need only the proportion of responses in one category and then we know the proportion of responses in the other.

2. <u>The three category case.</u>

(a) Consider now the case of 3 response categories, and still $\eta_v = 0$ for all v. This extended situation can be diagramed as before where now

$$\log(p_i^{(1)}/p_i^{(0)}) = l_{i1} = \sigma_{i1} = \mu_1 + \theta_{i1}$$
$$\log(p_i^{(2)}/p_i^{(0)}) = l_{i2} = \sigma_{i2} = \mu_2 + \theta_{i2}.$$

i	0	1	2
1	$p_1^{(0)}$	$p_1^{(1)}$	$p_1^{(2)}$
⋮	⋮	⋮	⋮
K	$p_K^{(0)}$	$p_K^{(1)}$	$p_K^{(2)}$

$p_i^{(1)}/p_i^{(0)}$	$p_i^{(2)}/p_i^{(0)}$
$p_1^{(1)}/p_1^{(0)}$	$p_1^{(2)}/p_1^{(0)}$
⋮	⋮
$p_K^{(1)}/p_K^{(0)}$	$p_K^{(2)}/p_K^{(0)}$

σ_{11}	σ_{12}
⋮	
σ_{K1}	σ_{K2}

μ_1	μ_2
θ_{11}	θ_{12}
⋮	⋮
θ_{K1}	θ_{K2}

$\mu_1+\mu_2$
δ_1
⋮
δ_K

For example:

0	1	2
·10	·20	·70
·20	·30	·50
·25	·50	·25
·30	·60	·10
·40	·50	·10

2.0	7.0
1.5	2.5
2.0	1.0
2.0	·33
1.25	·20

·69	1.95
·41	·92
·69	·00
·69	−1.11
·22	−1.38
$\Sigma\sigma_i$ 2.7	·38

·54	·08
·15	1.87
−·13	·84
·15	−·08
·15	−1.19
−·32	−1.44
$\Sigma\theta_i$ 0.0	0.0

·62
−·68
−·25
−·01
·35
·59
$\Sigma\hat{\delta}_i = 0$

Each item can have two parameters, namely θ_{i1} and θ_{i2}, and there are two category parameters, μ_1 and μ_2 which represent a kind of grand mean effect for each category relative to the first. Of course $\sum_{i=1}^{k}\theta_{i1}=0$ and $\sum_{i=1}^{k}\theta_{i2}=0$.

(b) Although we can estimate two relative parameters for each item, we may suppose that the items reflect a single dimension or trait, i.e. that θ_{i1} and θ_{i2} are both some functions of one item parameter δ_i. We may further suppose that each category works in the same way for all items and so factor θ_{i1} and θ_{i2} to

$$\theta_{i1} = \varphi_1(-\delta_i) \quad \text{and} \quad \theta_{i2} = \varphi_2(-\delta_i)$$

giving $$\sigma_{i1} = \mu_1 - \varphi_1\delta_i \quad \text{and} \quad \sigma_{i2} = \mu_2 - \varphi_2\delta_i .$$

The restrictions $\sum_{i=1}^{k}\theta_{i1}=0$ and $\sum_{i=1}^{k}\theta_{i2}=0$ will hold if $\sum_{i=1}^{k}\delta_i=0$

(c) The function φ is called a scoring function. One possibility for this function is that $\varphi_1=1$ and $\varphi_2=2$. Then

$$\sigma_{i1} = \mu_1 - \delta_i \quad \text{and} \quad \sigma_{i2} = \mu_2 - 2\delta_i .$$

(d) In the example we have

$$\hat{\sigma}_{i1} = \hat{\mu}_1 - \hat{\delta}_i \qquad \hat{\sigma}_{i2} = \hat{\mu}_2 - 2\hat{\delta}_i$$

$$\therefore \ \hat{\sigma}_{i1} + \hat{\sigma}_{i2} = (\hat{\mu}_1 + \hat{\mu}_2) - 3\hat{\delta}_1$$

for $i = 1$, i.e. $\cdot 69 + 1\cdot 95 = (\cdot 54 + \cdot 08) - 3\hat{\delta}_1$

i.e. $2\cdot 64 = \cdot 62 - 3\hat{\delta}_1$

i.e. $\delta_1 = [(2\cdot 64) - \cdot 62]/(-3) = -\cdot 68$

and for each i, $\hat{\delta}_i = (\hat{\sigma}_{i1} + \hat{\sigma}_{i2} - \cdot 62)/(-3)$.

These values are shown in the previous table.

3. The full model.

(a) We now work backwards and specify the probability model for the 3 category case.

$$\log\left(p_i^{(1)}/p_i^{(0)}\right) = \sigma_{i1} \Rightarrow p_i^{(1)}/p_i^{(0)} = e^{\sigma_{i1}}$$
$$\log\left(p_i^{(2)}/p_i^{(0)}\right) = \sigma_{i2} \Rightarrow p_i^{(2)}/p_i^{(0)} = e^{\sigma_{i2}}$$

$$\therefore p_i^{(1)} = p_i^{(0)} e^{\sigma_{i1}}, \quad p_i^{(2)} = p_i^{(0)} e^{\sigma_{i2}}$$
$$\Rightarrow p_i^{(1)} + p_i^{(2)} = p_i^{(0)}\left(e^{\sigma_{i1}} + e^{\sigma_{i2}}\right).$$

Adding $p_i^{(0)}$ to both sides we have

$$p_i^{(0)} + p_i^{(1)} + p_i^{(2)} = p_i^{(0)} + p_i^{(0)}\left(e^{\sigma_{i1}} + e^{\sigma_{i2}}\right)$$

i.e. $$1 = p_i^{(0)}\left[1 + e^{\sigma_{i1}} + e^{\sigma_{i2}}\right]$$

i.e. $$p_i^{(0)} = 1/\left(1 + e^{\sigma_{i1}} + e^{\sigma_{i2}}\right)$$

and the specification becomes

$$\boxed{\begin{aligned} p_i^{(0)} &= 1/\left(1+e^{\sigma_{i1}}+e^{\sigma_{i2}}\right) \\ p_i^{(1)} &= e^{\sigma_{i1}}/\left(1+e^{\sigma_{i1}}+e^{\sigma_{i2}}\right) \\ p_i^{(2)} &= e^{\sigma_{i2}}/\left(1+e^{\sigma_{i1}}+e^{\sigma_{i2}}\right) \end{aligned}} \qquad (**)$$

(b) The straightforward extension of the probability statements (**) from the probability statements (*) is evident.

We may now take one step further backwards and reintroduce the subject parameter for the three category case. We obtain

$$p_{vi}^{(0)} = 1/(1+e^{\eta_{v1}+\sigma_{i1}}+e^{\eta_{v2}+\sigma_{i2}})$$
$$p_{vi}^{(1)} = e^{\eta_{v1}+\sigma_{i1}}/(1+e^{\eta_{v1}+\sigma_{i1}}+e^{\eta_{v2}+\sigma_{i2}})$$
$$p_{vi}^{(2)} = e^{\eta_{v2}+\sigma_{i2}}/(1+e^{\eta_{v1}+\sigma_{i1}}+e^{\eta_{v2}+\sigma_{i2}}) .$$

These equations give the full specification with <u>two</u> subject parameters, η_{v1} and η_{v2}, and two item parameters σ_{i1} and σ_{i2}, for the 3 category case.

(c) By a similar reasoning with respect to the subject parameter η_v as we carried out with respect to the item parameters σ_i, we could first write

$$\eta_{v1} = \vartheta_1 + \varsigma_{v1} , \quad \eta_{v2} = \vartheta_2 + \varsigma_{v2}$$

and again suggest that the two parameters ς_{v1} and ς_{v2} are functions of a single trait denoted by β_v. We then get

$$\eta_{v1} = \vartheta_1 + \varphi_1\beta_v , \quad \eta_{v2} = \vartheta_2 + \varphi_2\beta_v$$

when the scoring function for the subjects is the same as for the items. If $\varphi_1 = 1$ and $\varphi_2 = 2$, then

$$\eta_{v1} = \vartheta_1 + \beta_v \quad \text{and} \quad \eta_{v2} = \vartheta_2 + 2\beta_v .$$

(d) With these specifications for σ_{i1}, σ_{i2} and η_{v1}, η_{v2}, we have

$$\eta_{v1}+\sigma_{i1} = \vartheta_1 + \beta_v + \mu_1 - \delta_i \quad \text{and} \quad \eta_{v2}+\sigma_{i2} = \vartheta_2+2\beta_v+\mu_2-2\delta_i$$
$$= (\vartheta_1+\mu_1) + 1(\beta_v - \delta_i) \quad \text{and} \quad = \vartheta_2+\mu_2+2(\beta_v-\delta_i)$$
$$= \alpha_1 + 1(\beta_v - \delta_i) \quad \text{and} \quad = \alpha_2+2(\beta_v-\delta_i)$$

where α_1 and α_2 are combined category parameters which can be interpreted as μ and ϑ were individually.

The probability specifications then become

$$p_{vi}^{(0)} = 1/(1 + e^{\alpha_1 + (\beta_v - \delta_i)} + e^{\alpha_2 + 2(\beta_v - \delta_i)})$$
$$p_{vi}^{(1)} = e^{\alpha_1 + (\beta_v - \delta_i)}/(1 + e^{\alpha_1 + (\beta_v - \delta_i)} + e^{\alpha_2 + 2(\beta_v - \delta_i)})$$
$$p_{vi}^{(2)} = e^{\alpha_2 + 2(\beta_v - \delta_i)}/(1 + e^{\alpha_1 + (\beta_v - \delta_i)} + e^{\alpha_2 + 2(\beta_v - \delta_i)})$$

(e) For a convenient symmetric notation for the probability specification, we note that we can multiply the numerator and denominator of each probability statement by a constant. We choose it as e^{α_0} giving

$$p_{vi}^{(0)} = e^{\alpha_0}/(e^{\alpha_0} + e^{(\alpha_0+\alpha_1)+(\beta_v - \delta_i)} + e^{(\alpha_0+\alpha_2)+2(\beta_v - \delta_i)})$$
$$p_{vi}^{(1)} = e^{(\alpha_0+\alpha_1)+(\beta_v - \delta_i)}/(e^{\alpha_0} + e^{(\alpha_0+\alpha_1)+(\beta_v - \delta_i)} + e^{(\alpha_0+\alpha_2)+2(\beta_v - \delta_i)})$$
$$p_{vi}^{(2)} = e^{(\alpha_0+\alpha_2)+(\beta_v - \delta_i)}/(e^{\alpha_0} + e^{(\alpha_0+\alpha_1)+(\beta_v - \delta_i)} + e^{(\alpha_0+\alpha_2)+2(\beta_v - \delta_i)})$$

We may now more simply specify

$$\gamma_0 = \alpha_0, \quad \gamma_1 = \alpha_0 + \alpha_1, \quad \gamma_2 = \alpha_0 + \alpha_2$$

giving

$$p_{vi}^{(0)} = e^{\gamma_0}/(e^{\gamma_0} + e^{\gamma_1 + (\beta_v - \delta_i)} + e^{\gamma_2 + 2(\beta_v - \delta_i)})$$
$$p_{vi}^{(1)} = e^{\gamma_1 + (\beta_v - \delta_i)}/(e^{\gamma_0} + e^{\gamma_1 + (\beta_v - \delta_i)} + e^{\gamma_2 + 2(\beta_v - \delta_i)})$$
$$p_{vi}^{(2)} = e^{\gamma_2 + 2(\beta_v - \delta_i)}/(e^{\gamma_0} + e^{\gamma_1 + (\beta_v - \delta_i)} + e^{\gamma_2 + 2(\beta_v - \delta_i)})$$

but now we need to specify a restriction among the γ's as we had to among the δ's – say $\sum_{k=0}^{2} \gamma_k = 0$.

(f) With say $m+1$ categories, $k = 0, \ldots, m$, the full model becomes simply

$$p_{vi}^{(k)} = p_{kvi} = p\{k_{vi} \mid \beta_v, \delta_i, \gamma\} = e^{\gamma_k + k(\beta_v - \delta_i)} \Big/ \sum_{k'=0}^{m} e^{\gamma_{k'} + k'(\beta_v - \delta_i)}$$

where $\sum_{k=0}^{m} \gamma_k = 0$ and $\sum_{i=1}^{L} \delta_i = 0$

4. Estimation and a test of fit

(a) We may go through the estimation of parameters and obtain a test of fit in the same way as with the dichotomous model. We simply have two extra equations. The equations are

$$\left.\begin{aligned} s_i - \sum_k \sum_v k\hat{p}_{kvi} &= 0 \\ t_v - \sum_k \sum_v k\hat{p}_{kvi} &= 0 \\ T_k - \sum_v \sum_i \hat{p}_{kvi} &= 0 \end{aligned}\right\} \text{ OR } \begin{aligned} s_i - \sum_k \sum_r n_r k \hat{p}_{kri} &= 0 \\ t - \sum_k \sum_i k \hat{p}_{kri} &= 0 \\ T_k - \sum_r \sum_i n_r \hat{p}_{kri} &= 0 \end{aligned}$$

with $\sum_{i=1}^{K} \hat{\delta}_i = 0$, $\sum_{k=0}^{m} \hat{\gamma}_k = 0$ where

$$s_i = \sum_{v=1}^{N} k_{vi}, \quad t_v = \sum_{i=1}^{K} k_{vi}, \quad T_k = \sum_{v=1}^{N} \sum_{i=1}^{K} (x_{vi}^{(k)}).$$

(b) For an overall fit test we have

$$\chi^2 \simeq \sum_{i=1}^{K} \sum_{g=1}^{G} \left(\sum_{v \in g} k_{vi} - \sum_{v \in g} E[k_{vi}] \right)^2 \Big/ \sum_{v \in g} V[k_{vi}]$$

where $E[k_{vi}] = \sum_{k=0}^{m} k\, p_{kvi}$

and $V[k_{vi}] = \left(\sum_{k=0}^{m} k^2 p_{kvi} \right) - \left(\sum_{k=0}^{m} k\, p_{kvi} \right)^2$,

and $g = 1, G$ are the adjacent class intervals into which subjects have been amalgamated.
$df \simeq (G-1)(K-1) - m$

5. Special Cases of the above model

(a) We have $p_{kvi} = e^{\gamma_k + k(\beta_v - \delta_i)} \Big/ \left(\sum_{k'=0}^{m} e^{\gamma_{k'} + k(\beta_v - \delta_i)} \right)$.

now we may write $\psi_k = e^{\gamma_k}$ giving

$$p_{kvi} = p\{k_{vi} \mid \beta_v, \delta_i, \underset{\sim}{\psi}\} = \psi_k \, e^{k(\beta_v - \delta_i)} \Big/ \left(\sum_{k=0}^{m} \psi_k \, e^{k(\beta_v - \delta_i)} \right).$$

(b)

Now ψ may also be specified in advance for certain response situations. For example, it may represent binomial coefficients i.e.

$$\psi_k = \binom{m}{k}$$

and then

$$p\{k_{vi} \mid \beta_v, \delta_i\} = \binom{m}{k} \frac{e^{k(\beta_v - \delta_i)}}{(1 + e^{\beta_v - \delta_i})^m}.$$

If ψ represents Poisson coefficients then $\psi_k = 1/k!$ and

$$p\{k_{vi} \mid \beta_v, \delta_i\} = \frac{1}{k!} \frac{e^{k(\beta_v - \delta_i)}}{e^{\beta_v - \delta_i}}.$$